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On non-perturbative contributions to vacuum energy in supersymmetric quantum mechanical models

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Abstract. Saturating the functional integral directly with instanton-anti-instanton-type fluctuations, the vacuum energies for a supersymmetric quantum mechanical system with (i) a double-well and (ii) a triple-well potential are studied. In the former, the vacuum energy is raised by these fluctuations, indicating spontaneous breakdown of supersymmetry. In the latter, the vacuum energy stays at zero, indicating that supersymmetry is not broken. In addition, the energy of the next level supersymmetric pair of states has been calculated for the triple-well case.

1. Introduction

Supersymmetry [1] as a possible fundamental symmetry between fermions and bosons has been studied for over a decade now. It appears to cure the notorious gauge hierarchy problem [2] in the grand unified theories [3, 4]. However, at ordinary energy scales, this symmetry is not exact in nature. Whereas perturbative quantum effects respect supersymmetry, it would be desirable if non-perturbative fluctuations were to break it. To this end, it is interesting to study the non-perturbative fluctuations such as instantons and anti-instantons [4-8].

In non-supersymmetric theories in $(0+1)$ dimensions, the instanton contributions to the vacuum energy have been discussed in [9-11]. For supersymmetric models in $(0+1)$ dimensions, the role of instantons has been studied in [6].

In this paper, we shall discuss instanton-type quantum fluctuations in supersymmetric quantum mechanical systems with (i) a double-well and (ii) a triple-well potential. In contrast to [6], where single-instanton-induced vacuum expectation values of supersymmetric generators were obtained, we shall saturate the functional integral directly with instanton-type fluctuations to obtain the vacuum energy. As is well known, single instantons or anti-instantons do not contribute to the vacuum functional integral because of the zero modes of the relevant fermion determinant obtained by integrating over fermionic degrees of freedom. However, in the background of an instanton-anti-instanton, this fermion determinant does not have any exact zero modes. Hence, such fluctuations may in general contribute to the vacuum energy. Here we develop a formalism to calculate these contributions. This can be easily generalised to study possible supersymmetry breaking in field-theoretic models in higher dimensions, whereas the formalism of [6] is not useful for such a calculation as the fermionic zero modes induce zero VEV for the supercharges in these models.

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The paper is organised as follows. In § 2 we present the supersymmetric quantum mechanical models in general. In § 3 we discuss the double-well potential. We find that the vacuum energy is shifted upwards due to instanton–anti-instanton fluctuations, thereby breaking supersymmetry spontaneously. The case of the triple-well potential is discussed in § 4. Here, in addition to instanton–anti-instanton effects, two-instanton and two-anti-instanton effects also contribute. However, the Hamiltonian matrix for the three lowest-lying states (which are independent linear combinations of the three classical ground states) has a zero and two equal positive eigenvalues. This implies that supersymmetry is not broken in this case. Finally, § 5 contains some concluding remarks.

2. Supersymmetric quantum mechanics

The Minkowskian action for a supersymmetric classical particle with anticommuting degrees of freedom may be written as

$$A_{\text{Mink}} = \frac{1}{2} \int dt[\dot{x}^2 - S^2(x) + i\psi^T\dot{\psi} - S'(x)\psi^T\sigma_2\psi] \tag{2.1}$$

where ψ is a two-component anticommuting variable

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \{\psi_i, \psi_j\} = 0 \tag{2.2}$$

and σ_2 is the Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

This action is invariant under the supersymmetric transformations

$$\delta_\varepsilon x = \varepsilon^T\sigma_2\psi \qquad \delta_\varepsilon\psi = (-i\sigma_2\dot{x} - S(x))\varepsilon$$
$$\delta_\varepsilon\psi^T = \varepsilon^T(i\sigma_2\dot{x} - S(x)).$$

We shall be studying the following two specific cases of this action:

(i) double-well potential

$$V \equiv \tfrac{1}{2}S^2 = \tfrac{1}{4}\lambda(x^2 - m^2/\lambda)^2 \tag{2.3}$$

(ii) triple-well potential

$$V \equiv \tfrac{1}{2}S^2 = (\lambda^2x^2/2m^2)(m^2/\lambda - x^2)^2. \tag{2.4}$$

In the former case, we have two classical ground states, denoted by $|\pm\rangle$, corresponding to $x = \pm m/\sqrt{\lambda}$ and $\psi = 0$. In the latter case, we have three classical ground states, denoted by $|\pm\rangle$, $|0\rangle$, corresponding to $x = \pm m/\sqrt{\lambda}$, 0 and $\psi = 0$. The classical vacuum energy for all these ground states is zero as dictated by supersymmetry. Perturbative quantum fluctuations around any one of these ground states do not change its energy. However, non-perturbative vacuum fluctuations which induce quantum mechanical tunnelling between various ground states may contribute to the vacuum energy.

3. Double-well potential

Let us study the effect of non-pertrubative vacuum fluctuations such as instantons and anti-instantons in the case of the double-well potential. Instantons and anti-instantons are the solutions of the Euclidean equations of motion implied by the Euclidean action

$$A_{\mathrm{E}}=\tfrac{1}{2}\int \mathrm{d}t[\dot{x}^2+S^2(x)+\psi^{\mathrm{T}}\dot{\psi}+S'(x)\psi^{\mathrm{T}}\sigma_2\psi] \tag{3.1}$$

and are given for the double-well potential (2.3) by

$$\begin{aligned} x_{\mathrm{I}}(t-t_1)&=\frac{m}{\sqrt{\lambda}}\tanh\left(\frac{m(t-t_1)}{\sqrt{2}}\right) \qquad && \psi_i=0 \\ x_{\bar{\mathrm{I}}}(t-t_2)&=-\frac{m}{\sqrt{\lambda}}\tanh\left(\frac{m(t-t_2)}{\sqrt{2}}\right) \qquad && \psi_i=0 \end{aligned} \tag{3.2}$$

respectively [10] (for a recent review of instanton physics, see [11]). These satisfy the linearised equations of motion $\dot{x}_{\mathrm{I}}=-S(x_{\mathrm{I}})$, $\dot{x}_{\bar{\mathrm{I}}}=S(x_{\bar{\mathrm{I}}})$, respectively. The classical action for both of them is the same:

$$A_0=2\sqrt{2}m^3/3\lambda. \tag{3.3}$$

To see the effect of these quantum fluctuations, we shall evaluate the following tunnelling amplitudes:

$$\langle\pm|\exp(-HT/\hbar)|\mp\rangle=\int_{x(-T/2)=\mp m/\sqrt{\lambda}}^{x(T/2)=\pm m/\sqrt{\lambda}}[\mathrm{d}x(t)]\,\mathrm{d}\psi_1(t)[\mathrm{d}\psi_2(t)]\exp(-A_{\mathrm{E}}/\hbar) \tag{3.4}$$

where T is the length of the large time box.

3.1. *Single-instanton (or single-anti-instanton) fluctuations*

An instanton contribution to the functional integral (3.4) can be obtained by expanding around the instanton:

$$x(t)=x_{\mathrm{I}}(t)+y(t) \tag{3.5}$$

which yields

$$\begin{aligned}\langle+|\exp(-HT/\hbar)|-\rangle_{x_{\mathrm{I}}}&=\exp(-A_0/\hbar)\int_{y(-T/2)=0}^{y(T/2)=0}[\mathrm{d}y(t)]\,\mathrm{d}\psi_1(t)[\mathrm{d}\psi_2(t)]\\ &\quad\times\exp\left[-\frac{1}{2\hbar}yD_{\mathrm{B}}y-\frac{1}{2\hbar}\psi^{\mathrm{T}}D_{\mathrm{F}}\psi\right]\end{aligned} \tag{3.6}$$

where

$$\begin{aligned} D_{\mathrm{B}}[x_{\mathrm{I}}]&\equiv-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+(S'^2+S''S)_{x_{\mathrm{I}}}=-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+m^2\left[3\tanh^2\left(\frac{m(t-t_1)}{\sqrt{2}}\right)-1\right]\\ D_{\mathrm{F}}[x_{\mathrm{I}}]&\equiv\frac{\mathrm{d}}{\mathrm{d}t}+\sigma_2S'(x_{\mathrm{I}})=\frac{\mathrm{d}}{\mathrm{d}t}+\sigma_2\sqrt{2}m\tanh\left(\frac{m(t-t_1)}{\sqrt{2}}\right).\end{aligned} \tag{3.7}$$

Naive integration over the bosonic fluctuations $y(t)$ yields $\det^{-1/2}D_{\mathrm{B}}$. However, this operator D_{B} has a zero eigenvalue corresponding to the invariance of the Euclidean action under the translation of the instanton location t_1. This has to be treated by the collective coordinate method, which yields an integration over the instanton location t_1 multiplied by the corresponding Jacobian factor $(A_0/2\pi\hbar)^{1/2}$.

Integration over the fermionic fluctuations yields $\det^{1/2} D_F$ in (3.6). This fermionic operator $D_F[x_I]$ also has a zero mode

$$\begin{aligned}\psi_0^{(+)}(t-t_1) &= \sqrt{\frac{\hbar}{A_0}}\, \dot{x}_I(t-t_1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\varepsilon_0^{(+)} \\ &= \sqrt{\frac{\hbar}{A_0}}\frac{m^2}{\sqrt{2\lambda}}\,\mathrm{sech}^2\!\left(\frac{m(t-t_1)}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\varepsilon_0^{(+)} \\ &\equiv \sqrt{\hbar}\,\chi^{(+)}(t-t_1)\varepsilon_0^{(+)}\end{aligned} \tag{3.8}$$

and the corresponding fermionic zero mode for the anti-instanton operator $D_F[x_{\bar{I}}]$ is

$$\begin{aligned}\psi_0^{(-)}(t-t_2) &= \sqrt{\frac{\hbar}{A_0}}\, \dot{x}_{\bar{I}}(t-t_2)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}\varepsilon_0^{(-)} \\ &= \sqrt{\frac{\hbar}{A_0}}\frac{m^2}{\sqrt{2\lambda}}\,\mathrm{sech}^2\!\left(\frac{m(t-t_2)}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}\varepsilon_0^{(-)} \\ &\equiv \sqrt{\hbar}\,\chi^{(-)}(t-t_2)\varepsilon_0^{(-)}.\end{aligned} \tag{3.9}$$

Here $\varepsilon_0^{(\pm)}$ are the anticommuting collective coordinates corresponding to these zero modes.

The functional integral (3.6) can now be represented as

$$\langle +|\exp(-HT/\hbar)|-\rangle_{x_I} = \exp(-A_0/\hbar)\int \mathrm{d}t_1\left(\frac{A_0}{2\pi\hbar}\right)^{1/2} K_{x_I}\int \mathrm{d}\varepsilon_0^{(+)} \tag{3.10}$$

with

$$K_{x_I} \equiv \left(\frac{\det' D_F[x_I]}{\det' D_B[x_I]}\right)^{1/2} \tag{3.11}$$

where primes denote that the zero modes have been factored out. Writing

$\det' D_F[x_I] = \{\det'[-\mathrm{d}^2/\mathrm{d}t^2 + (S'^2 + S''S)_{x_I}]\det[-\mathrm{d}^2/\mathrm{d}t^2 + (S'^2 - S''S)_{x_I}]\}^{1/2}$

we have

$$K_{x_I} = \left(\frac{\det \tilde{D}_F}{\det' D_B}\right)^{1/4}_{x_I} \equiv \left(\frac{\det[-\mathrm{d}^2/\mathrm{d}t^2 + (S'^2 - S''S)_{x_I}]}{\det'[-\mathrm{d}^2/\mathrm{d}t^2 + (S'^2 + S''S)_{x_I}]}\right)^{1/4} \tag{3.12}$$

where the operator in the numerator does not have a zero mode for the boundary conditions of (3.6) and hence is not primed. This determinantal factor can be evaluated by using the technique presented in the review of Gelfand and Yaglom [10-12] in a large box of length T:

$$K_{x_I} = E_0^{1/2}\left(\frac{\bar{M}_I(T/2)M_I(-T/2)}{\bar{N}_I(T/2)N_I(-T/2)}\right)^{1/4} \tag{3.13}$$

where

$$D_B[x_I]N_I = 0 \qquad \tilde{D}_F[x_I]M_I = 0 \tag{3.14}$$

and

$$\bar{N}_I(t) = N_I(t)\int_{-T/2}^{t}\frac{\mathrm{d}t}{N_I^2(t)} \qquad \bar{M}_I(t) = M_I(t)\int_{-T/2}^{t}\frac{\mathrm{d}t}{M_I^2(t)}. \tag{3.15}$$

In (3.13) the expression in parentheses represents det $\tilde{D}_F/\det D_B$, where det D_B contains E_0^2 which is the would-be zero mode when $T\to\infty$. This has been divided out to get the primed determinant. This would-be zero mode (exact in the limit $T\to\infty$) can be calculated as in [10]:

$$E_0^2 = \bar{N}_I(T/2)\left(\bar{N}_I(T/2)\int_{-T/2}^{T/2} dt\, N_I(t)\bar{N}_I(t) - N_I(T/2)\int_{-T/2}^{T/2} dt\, \bar{N}_I^2(t)\right)^{-1}. \tag{3.16}$$

Taking

$$N_I(t-t_1) = \text{sech}^2\left(\frac{m(t-t_1)}{\sqrt{2}}\right) \qquad M_I(t-t_1) = \cosh^2\left(\frac{m(t-t_1)}{\sqrt{2}}\right) \tag{3.17}$$

and the limit $T\to\infty$, equations (3.13) and (3.16) yield the determinantal factor to be

$$K_{x_I} = (2\sqrt{2}m)^{1/2}. \tag{3.18}$$

It is interesing to notice that we do not have an exact matching of Bose and Fermi non-zero eigenmodes here [13]. This is unlike the case of supersymmetric Yang-Mills theory in (3+1) dimensions where this ratio would be 1 due to exact Bose-Fermi cancellations [14].

Inserting (3.18) in (3.10), we can finally write the single-instanton contribution to the vacuum functional integral as

$$\langle +|\exp(-HT/\hbar)|-\rangle_{x_I} = \exp(-A_0/\hbar)\left(\frac{A_0}{2\pi\hbar}\right)^{1/2}(2\sqrt{2}m)^{1/2}\int dt_1\int d\varepsilon_0^{(+)}. \tag{3.19}$$

Similarly, the single-anti-instanton contribution can be written as

$$\langle -|\exp(-HT/\hbar)|+\rangle_{x_{\bar{I}}} = \exp(-A_0/\hbar)\left(\frac{A_0}{2\pi\hbar}\right)^{1/2}(2\sqrt{2}m)^{1/2}\int dt_2\int d\varepsilon_0^{(-)}. \tag{3.20}$$

Note that $K_{x_I} = K_{x_{\bar{I}}}$. $\varepsilon_0^{(-)}$ is the collective coordinate corresponding to the fermionic zero mode in the anti-instanton background.

As is obvious, because of the fermionic integration (over $\varepsilon_0^{(+)}$ and $\varepsilon_0^{(-)}$) both (3.19) and (3.20) are exactly zero. Hence, single instantons or anti-instantons do not induce any quantum tunnelling. In fact, in general, any number of instantons and anti-instantons does not affect quantum tunnelling as long as there is an excess of one instanton or anti-instanton (kink number $=\pm 1$). However, topologically trivial (kink number $=0$) configurations containing an equal number of instantons and anti-instantons do contribute to the vacuum functional integral. This is because there are no exact fermionic zero modes for such configurations. A pair consisting of an instanton and an anti-instanton would give the lowest-order effect from such vacuum fluctuations.

3.2. Well separated instanton–anti-instanton

An instanton–anti-instanton configuration is depicted in figure 1, with the instanton located at t_1 and the anti-instanton located at t_2:

$$x_{I\bar{I}}(t;\, t_1, t_2) = \begin{cases} x_I(t-t_1) & -T/2 < t \leqslant \alpha \\ x_{\bar{I}}(t-t_2) & \alpha \leqslant t < T/2 \end{cases} \tag{3.21}$$

where $\alpha = (t_1+t_2)/2$. This configuration is not a stationary point of the Euclidean action, but for infinite separation, $t_2 - t_1 \equiv \beta \to T \to \infty$, it approaches a stationary point.

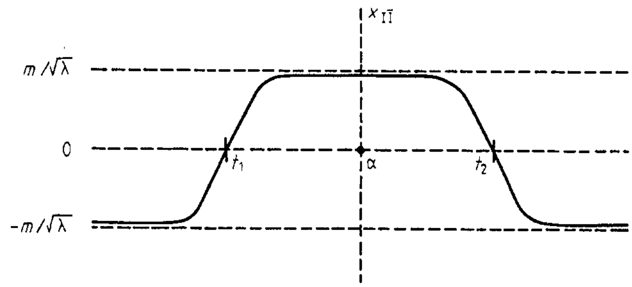

Figure 1. An instanton-anti-instanton configuration for the double-well case.

Under local translations of the instanton and anti-instanton, t_1, t_2, the action changes by an exponentially small amount and hence there are two approximate translational symmetries.

The action for this instanton-anti-instanton configuration can be written as

$$A_E[x_{I\bar{I}}] = \frac{1}{2}\int_{-T/2}^{T/2} dt[\dot{x}_{I\bar{I}}^2 + S^2(x_{I\bar{I}})]$$
$$= \int_{-T/2}^{\alpha} dt\, \dot{x}_I^2(t-t_1) + \int_{\alpha}^{T/2} dt\, \dot{x}_{\bar{I}}^2(t-t_2). \tag{3.22}$$

It is convenient to shift the integration variable from $t \to t+t_1$ and $t \to t+t_2$ in these two terms respectively. Then, in the approximation of large separation, $\beta = t_2 - t_1 \gg \sqrt{2}/m$, (3.22) can be written for $T \to \infty$ as

$$A_E[x_{I\bar{I}}] = \int_{-T/2}^{\beta/2} dt\, \dot{x}_I^2(t) + \int_{-\beta/2}^{T/2} dt\, \dot{x}_{\bar{I}}^2(t)$$
$$\simeq 2A_0 + A_{\text{int}}(\beta) \tag{3.23}$$

where

$$A_{\text{int}}(\beta) = \frac{3}{2} A_0\left[\tanh\left(\frac{m\beta}{2\sqrt{2}}\right) - \frac{1}{3}\tanh^3\left(\frac{m\beta}{2\sqrt{2}}\right) - \frac{2}{3}\right].$$

Here A_0 is the single-instanton action (3.3). As expected, for infinite separation, $\beta \to T \to \infty$, the interaction action $A_{\text{int}}(\beta)$ goes to zero.

In order to obtain the contribution of this fluctuation $x_{I\bar{I}}(t;\, t_1, t_2)$ to the functional integral for $\langle -|\exp(-HT/\hbar)|-\rangle$, we expand the action about this configuration. Corresponding to the two approximate translational zero modes, we introduce the integration over the two collective coordinates t_1 and t_2, with a Jacobian factor $(A_0/2\pi\hbar)^{1/2}$ for each one of them. We shall also evaluate the fermion determinant $K_0(t_2 - t_1)$ in the subspace of the fermionic zero modes (3.8) and (3.9) separately. The rest of the functional integral will be denoted by $K(\beta)$:

$$\langle -|\exp(-HT/\hbar)|-\rangle_{x_{I\bar{I}}} = \int_{x(-T/2)=-m/\sqrt{\lambda}}^{x(T/2)=m/\sqrt{\lambda}} [dx(t)]\, d\psi_1(t)[d\psi_2(t)] \exp(-A_E/\hbar)$$
$$\simeq \left(\frac{A_0}{2\pi\hbar}\right)\int_{-T/2}^{T/2} dt_2 \int_{-T/2}^{t_2-\beta_0} dt_1\, K(\beta)K_0(\beta)\exp\left(-\frac{2A_0}{\hbar} - \frac{A_{\text{int}}(\beta)}{\hbar}\right) \tag{3.24}$$

for large separation $\beta \gg \sqrt{2}/m$. Here $\beta_0 \gg \sqrt{2}/m$ is the minimal distance up to which our approximation would be valid. In order to avoid double counting, such a minimal separation is needed to distinguish the quantum fluctuations in the instanton-anti-instanton sector from those in the vacuum sector. In other words, only the non-perturbative contribution of the instanton-anti-instanton has to be considered. This is the contribution of the far-separated configuration. The almost overlapping instanton-anti-instanton configuration is taken into account when quantum fluctuations around the vacuum are calculated. These are perturbative contributions which do not break supersymmetry.

In this large separation region the instanton-anti-instanton configuration is almost a stationary point of the action, and $K(\beta)$ is simply the product of the instanton and anti-instanton non-zero modes determinantal factors of equation (3.18), $K(\beta) \simeq K_{x_{\rm I}} K_{x_{\bar{\rm I}}} = 2\sqrt{2m}$ $(\beta \gg \sqrt{2}/m)$. To estimate the minimal distance, β_0, we note that the approximation is valid provided the interaction action, $A_{\rm int}(\beta)$, is only a perturbation compared to A_0, i.e. $|A_{\rm int}(\beta)| \lesssim \hbar \ll A_0$. In this range we can neglect $A_{\rm int}(\beta)$ in the exponential.

Next, let us evaluate the fermion determinant in the subspace of zero modes given in (3.8) and (3.9):

$$
\begin{aligned}
K_0(\beta) = \int \mathrm{d}\varepsilon_0^{(-)}\,\mathrm{d}\varepsilon_0^{(+)} \exp\bigg\{-\frac{1}{2}\int \mathrm{d}t\bigg[\chi^{(-)\mathrm{T}}\bigg(\frac{\mathrm{d}}{\mathrm{d}t} + S'(x_{\rm I\bar{I}})\bigg)\chi^{(+)}\varepsilon_0^{(-)}\varepsilon_0^{(+)} \\
+ \chi^{(+)\mathrm{T}}\bigg(\frac{\mathrm{d}}{\mathrm{d}t} - S'(x_{\rm I\bar{I}})\bigg)\chi^{(-)}\varepsilon_0^{(+)}\varepsilon_0^{(-)}\bigg]\bigg\} \\
= \tfrac{1}{2}\int \mathrm{d}t\bigg[\chi^{(-)\mathrm{T}}\bigg(\frac{\mathrm{d}}{\mathrm{d}t} + S'(x_{\rm I\bar{I}})\bigg)\chi^{(+)} - \chi^{(+)\mathrm{T}}\bigg(\frac{\mathrm{d}}{\mathrm{d}t} - S'(x_{\rm I\bar{I}})\bigg)\chi^{(-)}\bigg].
\end{aligned}
\tag{3.25}
$$

We break the time integral from $-T/2$ to α and from α to $T/2$. Integration by parts and use of the equations satisfied by $\chi^{(+)}$ and $\chi^{(-)}$,

$$
\begin{aligned}
&[\mathrm{d}/\mathrm{d}t + S'(x_{\rm I}(t-t_1))]\chi^{(+)}(t-t_1) = 0 \qquad t<\alpha \\
&[\mathrm{d}/\mathrm{d}t - S'(x_{\bar{\rm I}}(t-t_2))]\chi^{(-)}(t-t_2) = 0 \qquad t>\alpha
\end{aligned}
\tag{3.26}
$$

yields for large separation and large T:

$$
K_0(\beta) \simeq -(3m/4\sqrt{2})\,\mathrm{sech}^4(m\beta/2\sqrt{2}). \tag{3.27}
$$

Notice that for $\beta \to T \to \infty$, $K_0 \to 0$ as expected, because in this limit the fermionic zero modes become exact.

Inserting (3.27) into (3.24) and performing the integrations, we find

$$
\langle -|\exp(-HT/\hbar)|-\rangle_{x_{\rm I\bar{I}}} \simeq \frac{A_0}{2\pi\hbar}\exp(-2A_0/\hbar)\,\frac{T}{2}\,2\sqrt{2}m\,\frac{A_{\rm int}(\beta_0)}{A_0} \tag{3.28}
$$

where $K(\beta) = 2\sqrt{2}m$. As argued above the minimal separation, β_0, is given by $A_{\rm int}(\beta_0) = -\hbar$, so we have

$$
\langle -|\exp(-HT/\hbar)|-\rangle_{x_{\rm I\bar{I}}} \simeq -\frac{mT}{\pi\sqrt{2}}\exp(-2A_0/\hbar)[1 + \mathrm{O}(\hbar/A_0)] \tag{3.29}
$$

where higher-order corrections in $\hbar/A_0$ were neglected. Those arise from corrections to $K(\beta)$ and the perturbative expansion around the instanton-anti-instanton. Note that $K(T) - K(\beta_0) \simeq \mathrm{O}(\exp(-m\beta_0/\sqrt{2}]) \simeq \mathrm{O}(\hbar/A_0)$, which is in agreement with our approximation.

Similar calculations could be done for the contribution of the anti-instanton-instanton contribution to $\langle +|\exp(-HT/\hbar)|+\rangle$ with the same result as given on the right-hand side of (3.29).

From these, we notice that we have two ground states in this model with equal energies

$$\langle +|H|+\rangle = \langle -|H|-\rangle \simeq (m\hbar/\pi\sqrt{2})\exp(-2A_0/\hbar) \tag{3.30}$$

up to the lowest order and supersymmetry is spontaneously broken. This result is in accordance with that of [6] where the ground-state energies were obtained by calculating the VEV of the supercharges in a background of instantons. Here, on the other hand, we calculate the vacuum energy directly by saturating the functional integral with an instanton–anti-instanton configuration. Note that had we integrated over β in (3.24) without ignoring $A_{\text{int}}(\beta)$ in the exponential, the same result would be found provided β_0 is determined by $A_{\text{int}}(\beta_0) = -\hbar \ln 2$. Here again $\|A_{\text{int}}(\beta_0)| \simeq O(\hbar) \ll A_0$ as argued above.

4. Triple-well potential

In this case, there is a general argument due to Witten [4] that supersymmetry is not broken by quantum effects. He has argued the existence of a normalisable zero-energy ground state from general principles. In the following, we shall demonstrate that, indeed, instanton–anti-instanton, two-instanton and two-anti-instanton effects conspire to leave a zero-energy ground state. We shall also calculate the energy of the next-level excited states.

The potential is given by the expression in (2.4). This model admits two types of instantons [15]:

$$\begin{aligned}
&\text{(i)}\quad \dot{x} = -S(x) \qquad x = x_{\text{I}}(t-t_0) \equiv \frac{m}{\sqrt{\lambda}}\left(\frac{1+\tanh[m(t-t_0)]}{2}\right)^{1/2}\\
&\text{(ii)}\quad \dot{x} = S(x) \qquad x = y_{\text{I}}(t-t_0) \equiv -\frac{m}{\sqrt{\lambda}}\left(\frac{1-\tanh[m(t-t_0)]}{2}\right)^{1/2}
\end{aligned} \tag{4.1}$$

which interpolate between the classical ground states $x=0$ as $t\to-\infty$ and $x=m/\sqrt{\lambda}$ as $t\to\infty$ (x_{I}), or $x=-m\sqrt{\lambda}$, as $t\to-\infty$ and $x=0$ as $t\to\infty(y_{\text{I}})$. There are also two types of anti-instantons [15]:

$$\begin{aligned}
&\text{(iii)}\quad \dot{x} = S(x) \qquad x = x_{\bar{\text{I}}}(t-t_0) \equiv \frac{m}{\sqrt{\lambda}}\left(\frac{1-\tanh[m(t-t_0)]}{2}\right)^{1/2}\\
&\text{(iv)}\quad \dot{x} = -S(x) \qquad x = y_{\bar{\text{I}}}(t-t_0) \equiv -\frac{m}{\sqrt{\lambda}}\left(\frac{1+\tanh[m(t-t_0)]}{2}\right)^{1/2}
\end{aligned} \tag{4.2}$$

which interpolate between the classical ground states $x=m/\sqrt{\lambda}$ as $t\to-\infty$ and $x=0$ as $t\to\infty(x_{\bar{\text{I}}})$, or $x=0$ as $t\to-\infty$ and $x=-m/\sqrt{\lambda}$ as $t\to\infty(y_{\bar{\text{I}}})$.

The action for all these instantons and anti-instantons is the same:

$$A_0 = m^3/4\lambda. \tag{4.3}$$

4.1. Single-instanton (anti-instanton) contribution

As in the double-well case, the single-instanton (or single-anti-instanton) contribution to the vacuum functional integral is completely suppressed due to the fermionic zero modes. These zero modes are the normalisable solutions of

$$[\mathrm{d}/\mathrm{d}t + \sigma_2 S'(x)]\psi = 0 \tag{4.4}$$

where x is any one of the configurations x_{I}, $x_{\bar{\mathrm{I}}}$, y_{I}, $y_{\bar{\mathrm{I}}}$ listed in (4.1) and (4.2) and $S(x)$ is as defined in (2.4). These zero modes can be written as

$$\begin{aligned}
&\text{(i)}\quad x_{\mathrm{I}} && \psi_0^{(+)}=\sqrt{\frac{\hbar}{A_0}\frac{m^2}{2\sqrt{2}\lambda}}\exp[-m(t-t_0)/2]\operatorname{sech}^{3/2}[m(t-t_0)]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\varepsilon_0^{(+)}\\
&\text{(ii)}\quad y_{\mathrm{I}} && \psi_0^{(-)}=\sqrt{\frac{\hbar}{A_0}\frac{m^2}{2\sqrt{2}\lambda}}\exp[m(t-t_0)/2]\operatorname{sech}^{3/2}[m(t-t_0)]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}\eta_0^{(-)}\\
&\text{(iii)}\quad x_{\bar{\mathrm{I}}} && \psi_0^{(-)}=\sqrt{\frac{\hbar}{A_0}\frac{m^2}{2\sqrt{2}\lambda}}\exp[m(t-t_0)/2]\operatorname{sech}^{3/2}[m(t-t_0)]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}\varepsilon_0^{(-)}\\
&\text{(iv)}\quad y_{\bar{\mathrm{I}}} && \psi_0^{(+)}=\sqrt{\frac{\hbar}{A_0}\frac{m^2}{2\sqrt{2}\lambda}}\exp[-m(t-t_0)/2]\operatorname{sech}^{3/2}[m(t-t_0)]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\eta_0^{(+)}.
\end{aligned}\tag{4.5}$$

Similarly to (3.10), the various tunnelling amplitudes ($|-\rangle\to|0\rangle$, $|0\rangle\to|+\rangle$, etc) can be expressed in terms of integration over the collective coordinates representing the fermionic and bosonic zero modes and the non-zero-mode determinantal factor of the form (3.13)-(3.16). Here we have

$$\begin{aligned}
K_{y_{\mathrm{I}}}=K_{x_{\bar{\mathrm{I}}}}&=\left(\frac{\det\tilde{D}_{\mathrm{F}}}{\det' D_{\mathrm{B}}}\right)^{1/4}_{y_{\mathrm{I}},x_{\bar{\mathrm{I}}}}\\
&=\left(\frac{\det\{-\mathrm{d}^2/\mathrm{d}\tau^2+\frac{1}{4}m^2[15\tanh^2(m\tau)-6\tanh(m\tau)-5]\}}{\det'\{-\mathrm{d}^2/\mathrm{d}\tau^2+\frac{1}{4}m^2[3\tanh^2(m\tau)-6\tanh(m\tau)+7]\}}\right)^{1/4}
\end{aligned}\tag{4.6}$$

and $K_{x_{\mathrm{I}}}=K_{y_{\bar{\mathrm{I}}}}$ is given by the same expression with $\tau\equiv t-t_0$ replaced by $-\tau\equiv-(t-t_0)$. In the formulae (3.13)-(3.16), the N and M functions are for the present case defined as follows:

$$\begin{aligned}
M_{y_{\mathrm{I}}}(\tau)&=M_{x_{\bar{\mathrm{I}}}}(\tau)=\exp(-m\tau/2)\cosh^{3/2}(m\tau)\\
N_{y_{\mathrm{I}}}(\tau)&=N_{x_{\bar{\mathrm{I}}}}(\tau)=\exp(m\tau/2)\operatorname{sech}^{3/2}(m\tau)
\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}
M_{x_{\mathrm{I}}}(\tau)&=M_{y_{\bar{\mathrm{I}}}}(\tau)=\exp(m\tau/2)\cosh^{3/2}(m\tau)\\
N_{x_{\mathrm{I}}}(\tau)&=N_{y_{\bar{\mathrm{I}}}}(\tau)=\exp(-m\tau/2)\operatorname{sech}^{3/2}(m\tau).
\end{aligned}\tag{4.8}$$

Using these, we obtain from (3.13)-(3.16) by a direct computation after taking the limit $T\to\infty$:

$$K_{x_{\mathrm{I}}}=K_{x_{\bar{\mathrm{I}}}}=K_{y_{\mathrm{I}}}=K_{y_{\bar{\mathrm{I}}}}=(2\sqrt{2}m)^{1/2}.\tag{4.9}$$

Since the single-instanton (or single-anti-instanton) effects are completely suppressed, the possible next-order contribution to the vacuum functional integral will again come from instanton-anti-instanton configurations. In this case, we also have to include two-instanton and two-anti-instanton contributions representing quantum tunnellings $|-\rangle$ to $|+\rangle$ and $|+\rangle$ to $|-\rangle$. All these we consider in the next subsection.

4.2. Instanton-anti-instanton, two-instanton and two-anti-instanton contributions

We now shall evaluate the contribution of the fluctuations of the type depicted in figure 2. Figures 2(a) and 2(b) contribute to the matrix element $\langle 0|\exp(-HT/\hbar)|0\rangle$, figure 2($c$) contributes to $\langle -|\exp(-HT/\hbar)|-\rangle$ and figure 2(d) to $\langle +|\exp(-HT/\hbar)|+\rangle$.

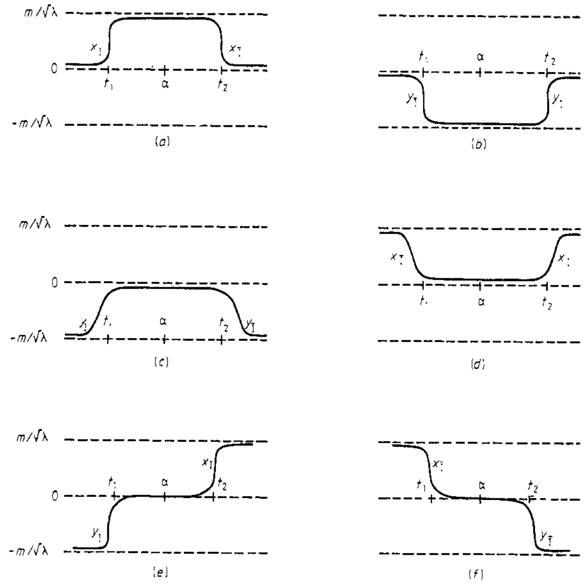

Figure 2. Configurations for the triple-well case: (*a*) instanton-anti-instanton; (*b*) anti-instanton-instanton; (*c*) instanton-anti-instanton; (*d*) anti-instanton-instanton; (*e*) two-instanton; (*f*) two-anti-instanton.

The instanton-instanton configuration of figure 2(*e*) contributes to the tunnelling amplitude $\langle +|\exp(-HT/\hbar)|-\rangle$ and the anti-instanton-anti-instanton fluctuation of figure 2(*f*) contributes to the tunnelling amplitude $\langle -|\exp(-HT/\hbar)|+\rangle$. In particular, the instanton-anti-instanton contribution of figure 2(*a*) can be written as

$$\langle 0|\exp(-HT/\hbar)|0\rangle_{x_I x_{\bar{I}}} = \left(\frac{A_0}{2\pi\hbar}\right)\int_{-T/2}^{T/2} \mathrm{d}t_2 \int_{-T/2}^{t_2-\beta_0} \mathrm{d}t_1\, K(\beta)K_0^{(1)}(\beta)\exp(-A[x_I, x_{\bar{I}}]/\hbar) \tag{4.10}$$

where the action for this configuration can be written as

$$\begin{aligned} A[x_I, x_{\bar{I}}] &= \frac{1}{2}\left[\int_{-T/2}^{\alpha} + \int_{\alpha}^{T/2}\right] \mathrm{d}t(\dot{x}^2 + S^2) \\ &= \int_{-T/2}^{\beta/2} \mathrm{d}t\, \dot{x}_I^2(t) + \int_{-\beta/2}^{T/2} \mathrm{d}t\, \dot{x}_{\bar{I}}^2(t) \\ &\simeq 2A_0 + A_{\mathrm{int}}^{(1)}(\beta) \end{aligned} \tag{4.11}$$

with

$$A_{\mathrm{int}}^{(1)}(\beta) = A_0[\tanh(m\beta/2) + \tfrac{1}{2}\operatorname{sech}^2(m\beta/2) - 1]$$

for large separations $\beta = t_2 - t_1 \gg 2/m$. As earlier, $K_0^{(1)}(\beta)$ is the fermionic determinant evaluated in the subspace of the relevant fermionic zero modes listed in (4.5). As in the previous section (equations (3.25)-(3.27)), we can approximate this as follows:

$$K_0^{(1)}(\beta) \simeq -\tfrac{1}{2}m\exp(-m\beta/2)\operatorname{sech}^2(m\beta/2). \tag{4.12}$$

Again, for large separation, $\beta \to T \to \infty$, when the instanton-anti-instanton configuration becomes a stationary point of the action, the function $K(\beta)$ in (4.10) reduces to the product of the instanton and anti-instanton non-zero-mode determinantal factors given in (4.9), $K(T) \xrightarrow[T\to\infty]{} K_{x_I}K_{x_I} = 2\sqrt{2}\mathrm{m}$.

As argued above, the small separated configuration is included in the vacuum sector and the integrations are performed from a minimal distance $\beta_0 \ll 2/m$ given by $|A_{\rm int}(\beta_0)| \simeq \hbar \ll A_0$, up to infinity. In this range $A_{\rm int}$ in the exponential is negligible compared with A_0.

Inserting (4.11) and (4.12) into (4.10) and doing the integrations we have

$$\langle 0|\exp(-HT/\hbar)|0\rangle_{x_I x_{\bar{I}}} \simeq \frac{A_0}{2\pi\hbar} \exp(-2A_0/\hbar) \frac{T}{2} 2\sqrt{2}m \frac{A_{\rm int}(\beta_0)}{A_0}. \tag{4.13}$$

Thus for $A_{\rm int}(\beta_0) = -\hbar$ we find

$$\langle 0|\exp(-HT/\hbar)|0\rangle_{x_I x_{\bar{I}}} \simeq -\frac{mT}{\sqrt{2}\pi} \exp(-2A_0/\hbar). \tag{4.14}$$

By symmetry, the fluctuation depicted in figure 2(*b*) contributes the same amount

$$\langle 0|\exp(-HT/\hbar)|0\rangle_{y_{\bar{I}} y_I} \simeq -\frac{mT}{\sqrt{2}\pi} \exp(-2A_0/\hbar) \tag{4.15}$$

so that adding (4.14) and (4.15) yields the energy for this state to be

$$\langle 0|H|0\rangle_{x_I x_{\bar{I}} + y_{\bar{I}} y_I} \simeq \frac{\sqrt{2}m\hbar}{\pi} \exp(-2A_0/\hbar). \tag{4.16}$$

With regard to the fluctuations depicted in figures 2(*c*)–2(*f*), the calculations proceed exactly as above. The action for all these configurations is equal:

$$A[y_I, y_{\bar{I}}] = A[x_{\bar{I}}, x_I] = A[y_I, x_I] = A[x_{\bar{I}}, y_{\bar{I}}] \simeq 2A_0 + A^{(2)}_{\rm int}(\beta)$$

with

$$A^{(2)}_{\rm int}(\beta) = A_0[\tanh(m\beta/2) + \tfrac{1}{2}\tanh^2(m\beta/2) - \tfrac{3}{2}] \tag{4.17}$$

for large separations, $\beta \gg 2/m$. The fermionic determinants in the subspace of the relevant zero modes (4.5), in all these cases, are also equal and can be approximated by

$$K_0^{(2)}(\beta) \simeq -\tfrac{1}{2}m \exp(m\beta/2)\,\mathrm{sech}^3(m\beta/2) \tag{4.18}$$

for large separations. Now a rerun of the arguments presented above yields

$$\langle -|H|-\rangle_{y_I y_{\bar{I}}} = \langle +|H|+\rangle_{x_{\bar{I}} x_I} = \langle +|H|-\rangle_{y_I x_I} = \langle -|H|+\rangle_{x_{\bar{I}} y_{\bar{I}}} \simeq \frac{m\hbar}{\sqrt{2}\pi} \exp(-2A_0/\hbar) \tag{4.19}$$

in the lowest order.

Finally, the Hamiltonian matrix for the low-lying states can be written as

$$H = \begin{pmatrix} X & 0 & X \\ 0 & 2X & 0 \\ X & 0 & X \end{pmatrix} \qquad X \simeq \frac{m\hbar}{\sqrt{2}\pi} \exp(-2A_0/\hbar) \tag{4.20}$$

in the lowest order. This matrix has a zero eigenvalue and two equal non-zero eigenvalues, $(\sqrt{2}m\hbar/\pi)\exp(-2A_0/\hbar)$. Hence supersymmetry is *not* broken in this case in contrast to that of the double-well potential. This is in accordance with the arguments of [4, 6]. Further, the appearance of the next-lying states of equal energy is also as dictated by supersymmetry. Note that our formalism allows the calculation of the next-lying states, which is not the case with the formalism of [6].

5. Concluding remarks

By saturating the functional integral directly with instanton-anti-instanton, two-instanton and two-anti-instanton fluctuations, we have calculated the vacuum energy of a supersymmetric quantum mechanical particle moving in double-well and triple-well potentials. The vacuum energy in the former case does get raised from zero due to these quantum fluctuations, implying a spontaneous breakdown of supersymmetry. This vacuum energy has been calculated to the lowest order and the result is in agreement with that obtained in [6]. In the case of a triple-well potential, we find that there does survive a zero-energy ground state even when these types of fluctuations are included and therefore supersymmetry is not broken. We have also obtained the energy of the supersymmetric pair of the next-level states to the lowest order. They have the same energy in accordance with supersymmetry.

It is interesting to compare the models studied here with the two $N=2$ supersymmetric quantum mechanical models studied in [16], the first of which exhibits no supersymmetry breaking whereas in the second supersymmetry is broken. The breaking of supersymmetry in the second model is attributed to complex instanton solutions which induce tunnelling effects and shift the vacuum energy. Those solutions are missing in the first model. In both cases supersymmetry breaking was studied by calculating the vacuum energy for strong coupling and analysing its behaviour when the coupling is changed. The result has then been confirmed by studying the VEV of supercharges in a background of instantons. Alternatively, one can use the formalism presented in the present paper to calculate the vacuum energy by saturating the functional integral by the instanton-anti-instanton configuration. It is expected that the first model of [16] would exhibit the behaviour of the triple-well potential (equation (4.20)) whereas in the second the vacuum energy would be shifted as in equation (3.30) for the double well. One should only remember to take into account the complex instanton solutions of the second model of [16].

The method presented here is fairly general and can be used to explore the possible supersymmetry breaking in other models which admit an instanton type of vacuum fluctuations. It is particularly useful for field theoretic models, where due to the fermionic zero modes VEV of supercharges vanish in a background of instantons, and cannot therefore be used for the study of supersymmetry breaking effects.

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